

# A MATRIX AND ITS INVERSE: REVISITING MINIMAL RANK COMPLETIONS

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## Abstract

We revisit a formula that connects the minimal ranks of triangular parts of a matrix and its inverse and relate the result to structured rank matrices. We also address a generic minimal rank problem that was proposed by David Ingerman and Gilbert Strang.

**Key words:** minimal rank, matrix completion, nullity theorem, band matrix, semi-separable, quasi-separable.

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## 1 Introduction

In this paper we revisit the following result from [22]:

Let  $[(T_{ij})_{i,j=1}^n]^{-1} = (S_{ij})_{i,j=1}^n$  be block matrices with sizes that are compatible for multiplication. Other than the full matrix (which is of size  $N$ , say), none of the blocks need to be square. Then

$$\min \operatorname{rank} \begin{pmatrix} T_{11} & ? & \cdots & ? \\ T_{21} & T_{22} & \cdots & ? \\ \vdots & & \ddots & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{pmatrix} + \min \operatorname{rank} \begin{pmatrix} ? & ? & \cdots & ? \\ S_{21} & ? & \cdots & ? \\ \vdots & \ddots & \ddots & \vdots \\ S_{n1} & \cdots & S_{n,n-1} & ? \end{pmatrix} = N. \quad (1.1)$$

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With the recent interest in numerical algorithms that make effective use of matrices with certain rank structures (see, e.g., [4], [21], [18], [7], [9], and references therein), it seems appropriate to revisit this formula that captures many of the rank considerations that go into these algorithms. The nullity theorem due to [10] is a particular case. The papers [17] and [19] show the recent interest in the nullity theorem. It is our hope that this general formula (1.1) enhances the insight in rank structured matrices.

In addition, in Section 3 we will address the so-called "generic minimal rank problem". This problem was introduced by Professors Gilbert Strang and David Ingerman.

## 2 Minimal ranks of matrices and their inverses

Let us recall the notion of partial matrices and their minimal rank. Let  $\mathbb{F}$  be a field and let  $n, m, \nu_1, \dots, \nu_n, \mu_1, \dots, \mu_m$  be nonnegative integers. The *pattern* of specified entries in a partial matrix will be described by a set  $J \subset \{1, \dots, n\} \times \{1, \dots, m\}$ . A pattern  $K$  that is a subset of  $J$  will be called a *subpattern* of  $J$ . Let now  $A_{ij}, (i, j) \in J$ , be given matrices with entries in  $\mathbb{F}$  of size  $\nu_i \times \mu_j$ . We will allow  $\nu_i$  and  $\mu_j$  to equal 0. The collection of matrices  $\mathcal{A} = \{A_{ij}; (i, j) \in J\}$  is called a *partial block matrix with the pattern  $J$* . When all the blocks are of size  $1 \times 1$  (i.e.,  $\nu_i = \mu_j = 1$  for all  $i$  and  $j$ ), we will simply talk about a *partial matrix*. Clearly, any block matrix as above may be viewed as a partial matrix of size  $N \times M$  as well, where  $N = \nu_1 + \dots + \nu_n, M = \mu_1 + \dots + \mu_m$ . It will be convenient to represent partial block matrices in matrix format. As usual a question mark will represent an unknown block. For instance,  $\mathcal{A} = \{A_{ij} : 1 \leq j \leq i \leq n\}$  will be represented as

$$\mathcal{A} = \begin{pmatrix} A_{11} & ? & \dots & ? \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & ? \\ A_{n1} & \dots & \dots & A_{nn} \end{pmatrix}$$

Let a partial matrix  $\mathcal{A} = \{A_{ij}; (i, j) \in J\}$  be given. A block matrix  $B = (B_{ij})_{i=1, j=1}^n, m$  with  $B_{ij} \in \mathbb{F}^{\nu_i \times \mu_j}$  is called a *completion* of  $\mathcal{A}$  if  $B_{ij} = A_{ij}, (i, j) \in J$ . The *minimal rank* of  $\mathcal{A}$  (notation:  $\min \text{rank}(\mathcal{A})$ ) is defined by

$$\min \text{rank}(\mathcal{A}) = \min\{\text{rank } B : B \text{ is a completion of } \mathcal{A}\}.$$

The formula that connects the minimal ranks of triangular parts of a matrix and its inverse is the following. The result appeared originally in [22] (see also [24] and Chapter 5 of [23]).

**Theorem 2.1** [22] *Let  $T = (T_{ij})_{i,j=1}^n$  be an invertible block matrix with  $T_{ij}$  of size  $\nu_i \times \mu_j$ , where  $\nu_i \geq 0, \mu_j \geq 0$  and  $N = \nu_1 + \dots + \nu_n = \mu_1 + \dots + \mu_n$ . Put  $T^{-1} = (S_{ij})_{i,j=1}^n$*

where  $S_{ij}$  is of size  $\mu_i \times \nu_j$ . Then

$$\min \text{rank} \begin{pmatrix} T_{11} & ? & \cdots & ? \\ T_{21} & T_{22} & \cdots & ? \\ \vdots & & \ddots & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{pmatrix} + \min \text{rank} \begin{pmatrix} ? & ? & \cdots & ? \\ S_{21} & ? & \cdots & ? \\ \vdots & \ddots & \ddots & \vdots \\ S_{n1} & \cdots & S_{n,n-1} & ? \end{pmatrix} = N.$$

As we will see, one easily deduces from Theorem 2.1 that the inverse of an upper Hessenberg matrix has the lower triangular part of a rank 1 matrix. The strength of Theorem 2.1 lies in that one easily deduces a multitude of such results from it.

From the same paper [22] we would also like to recall the following result.

**Theorem 2.2** [22] *The partial matrix  $\mathcal{T} = \{T_{ij} : 1 \leq j \leq i \leq n\}$  has minimal rank*

$$\min \text{rank} \mathcal{T} = \sum_{i=1}^n \text{rank} \begin{pmatrix} T_{i1} & \cdots & T_{ii} \\ \vdots & & \vdots \\ T_{n1} & \cdots & T_{ni} \end{pmatrix} - \sum_{i=1}^{n-1} \text{rank} \begin{pmatrix} T_{i+1,1} & \cdots & T_{i+1,i} \\ \vdots & & \vdots \\ T_{n1} & \cdots & T_{ni} \end{pmatrix}.$$

For the  $2 \times 2$  case of Theorem 2.2 one needs to observe that the minimal rank of

$$\begin{pmatrix} T_{11} & ? \\ T_{21} & T_{22} \end{pmatrix}$$

will at least be the rank of  $\begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix}$  plus the minimal number of columns in  $T_{22}$  that together with the columns of  $T_{21}$  span the column space of  $\begin{pmatrix} T_{21} & T_{22} \end{pmatrix}$ . Once such a minimal set of columns in  $T_{22}$  has been identified, put any numbers on top of these columns. Now any other columns in  $T_{22}$  can be completed to be a linear combination of fully completed columns. Doing this leads to a completion of rank

$$\text{rank} \begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix} + \text{rank} \begin{pmatrix} T_{21} & T_{22} \end{pmatrix} - \text{rank} T_{21},$$

yielding the  $n = 2$  case of Theorem 2.2. The general case now follows easily by induction.

The proof of Theorem 2.1, which can be found in [22] is easily derived from Theorem 2.2 and the nullity theorem, which we recall now.

**Theorem 2.3** [10] *Consider*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}.$$

*Then  $\dim \ker C = \dim \ker R$ .*

**Proof.** Since  $CP = -DR$ ,  $P[\ker R] \subseteq \ker C$ . Likewise, since  $RA = -SC$ , we get  $A[\ker C] \subseteq \ker R$ . Consequently,

$$AP[\ker R] \subseteq A[\ker C] \subseteq \ker R.$$

Since  $AP + BR = I$ ,  $AP[\ker R] = \ker R$ , thus

$$A[\ker C] = \ker R.$$

This yields  $\dim \ker C \geq \dim \ker R$ . By reversing the roles of  $C$  and  $R$  one obtains also that  $\dim \ker R \geq \dim \ker C$ . This gives  $\dim \ker R = \dim \ker C$ , yielding the lemma.  $\square$

The nullity theorem is in fact the  $n = 2$  case of Theorem 2.1. Indeed, if

$$T^{-1} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}^{-1} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix},$$

we get from Theorem 2.1 that

$$\text{rank} \begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix} + \text{rank} \begin{pmatrix} T_{21} & T_{22} \end{pmatrix} - \text{rank} T_{21} + \text{rank} S_{21} = N. \quad (2.2)$$

As  $T$  is invertible we have that  $\begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix}$  and  $\begin{pmatrix} T_{21} & T_{22} \end{pmatrix}$  are full rank, so (2.2) gives

$$\mu_1 + \nu_2 - \text{rank} T_{21} + \text{rank} S_{21} = \mu_1 + \mu_2 = \nu_1 + \nu_2,$$

and thus

$$\nu_2 - \text{rank} T_{21} = \mu_2 - \text{rank} S_{21},$$

which is exactly Theorem 2.3.

To make the connection with some of the results in the literature we need the following proposition.

**Proposition 2.4** *Let  $\mathcal{T} = \{t_{ij} : 1 \leq j \leq i \leq n\}$  be a scalar valued partial matrix. Then  $\min \text{rank}(\mathcal{T}) = n$  if and only if  $t_{ii} \neq 0, i = 1, \dots, n$ , and  $t_{ij} = 0$  for  $i > j$ .*

**Proof.** The "if" part is immediate. For the only if part write

$$\min \text{rank} \mathcal{T} = \text{rank} \begin{pmatrix} t_{11} \\ \vdots \\ t_{n1} \end{pmatrix} + \sum_{i=2}^n s_i, \quad (2.3)$$

where

$$s_i = \text{rank} \begin{pmatrix} t_{i1} & \dots & t_{ii} \\ \vdots & & \vdots \\ t_{n1} & \dots & t_{ni} \end{pmatrix} - \text{rank} \begin{pmatrix} t_{i1} & \dots & t_{i,i-1} \\ \vdots & & \\ t_{n1} & \dots & t_{n,i-1} \end{pmatrix}.$$

All the terms in (2.3) are at the most 1, and as there are exactly  $n$  terms they need to all be equal to 1 for  $\min \text{rank}(\mathcal{T}) = n$  to be satisfied. But then  $s_n = 1$  implies  $t_{n1} = \dots = t_{n,n-1} = 0$  and  $t_{nn} \neq 0$ . Inductively, one can then show that  $s_k = 1$  implies  $t_{k1} = \dots = t_{k,k-1} = 0$  and  $t_{kk} \neq 0, k = n-1, \dots, 2$ . Finally the first column of  $\mathcal{T}$  needs to have rank 1. As  $t_{ij} = 0, j = 2, \dots, n$ , was already established we get that  $t_{11} \neq 0$ . This proves the result.  $\square$

We now easily obtain the following corollary, due to Asplund [1].

**Corollary 2.5** [1] *Let  $p \geq 0$  and  $A = (a_{ij})_{i,j=1}^N$  be an  $N \times N$  scalar matrix with inverse  $B = (b_{ij})_{i,j=1}^N$ . Then  $a_{ij} = 0$  for all  $i$  and  $j$  with  $j > i + p$ , and  $a_{ij} \neq 0, j = i + p$  if and only if there exist a  $N \times p$  matrix  $F$  and a  $p \times N$  matrix  $G$  so that  $b_{ij} = (FG)_{ij}, i < j + p$ . In particular, if  $p = 1$  (so  $A$  is lower Hessenberg) then  $b_{ij} = F_i G_j, 1 \leq i \leq j \leq N$ , where  $F_1, \dots, F_N, G_1, \dots, G_N$  are scalars.*

**Proof.** Let  $(S_{ij})_{i,j=N-p+1} = A$ , where  $S_{i1}$  is of size  $1 \times p, i = 1, \dots, n-p, S_{N-p+1,1}$  has size  $p \times p, S_{N-p+1,j}$  has size  $p \times 1, j = 2, \dots, N-p+1$ , and all the other  $S_{ij}$  are  $1 \times 1$ . Let  $B = (T_{ij})_{i,j=1}^{N-p+1}$  be partitioned accordingly. Then, it follows from (1.1) that

$$\min \text{rank} \begin{pmatrix} ? & ? & \dots & ? \\ S_{21} & ? & \dots & ? \\ \vdots & \ddots & \ddots & \vdots \\ S_{n1} & \dots & S_{n,n-1} & ? \end{pmatrix} = N - p$$

if and only if

$$\min \text{rank} \begin{pmatrix} T_{11} & ? & \dots & ? \\ T_{21} & T_{22} & \dots & ? \\ \vdots & & \ddots & \vdots \\ T_{n1} & T_{n2} & \dots & T_{nn} \end{pmatrix} = p.$$

Using Proposition 2.4 the result now follows.  $\square$

Corollary 2.5 shows that Theorem 2.1 is useful in the contexts of semi-separability and quasi-separability (see, e.g., [20] and [6] for an overview of these notions). In a similar way it is easy to deduce results by [3], [13], [14], [15], [16] and [8] from Theorem 2.1.

### 3 The generic minimal rank completion problem

Let  $r \in \mathbb{N}$ . We will call a pattern  $J \subseteq \{1, \dots, n\} \times \{1, \dots, m\}$   $r$ -sparse if for every  $K \subset \{1, \dots, n\}$  and every  $L \subseteq \{1, \dots, m\}$  with  $|K| = |L|$  we have that

$$|J \cap (K \times L)| \leq (2|K| - r)r.$$

Thus square submatrices of size  $N$  intersect the pattern in at most  $2Nr - r^2$  positions. The number  $2Nr - r^2$  comes from the situation where exactly  $r$  rows and  $r$  columns in an  $N \times N$  submatrix are prescribed.

Recently D. Ingerman and G. Strang observed that a 1-sparse matrix with nonzero entries has a completion of rank 1. We will give a proof of this fact below. This observation led Ingerman and Strang to ask: Is it true that one can "generically" complete a  $r$ -sparse partial matrix to a matrix of rank  $\leq r$ ? Of course, to be able to answer the question one needs to define "generically". The 1-sparse case suggests that it may suffice to require that all fully specified matrices are nonsingular. The following example, however, shows that this is not the right formulation of "generic".

**Example 3.1** Consider the matrix

$$A := \begin{pmatrix} 6 & 3 & x & 1 \\ 3 & 1 & 1 & y \\ z & 1 & 2 & 3 \\ 1 & w & 1 & 1 \end{pmatrix},$$

where  $x, y, z$  and  $w$  are the unknowns. Note that this partial matrix satisfies the requirements stated in the first paragraph. Furthermore, suppose that  $\text{rank} A = 2$ . Then we have that

$$\begin{pmatrix} 6 & 3 \\ 3 & 1 \end{pmatrix} - \begin{pmatrix} x & 1 \\ 1 & y \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} z & 1 \\ 1 & w \end{pmatrix} = 0,$$

and since the rank of the first term is 2, the second term must also have rank 2. Thus, we have that  $xy \neq 1$  and  $zw \neq 1$ . Next, we also have that

$$\begin{pmatrix} z & 1 \\ 1 & w \end{pmatrix} - \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x & 1 \\ 1 & y \end{pmatrix}^{-1} \begin{pmatrix} 6 & 3 \\ 3 & 1 \end{pmatrix} = 0.$$

Multiplying on both sides with  $xy - 1$ , the off-diagonal entries yield the following equations

$$xy - 6y - 3x + 10 = 0, \quad xy - 6y - 3x + 8 = 0.$$

These are not simultaneously solvable (as long as we are in a field where  $8 \neq 10$ ). It should be noted that this is a counterexample for any field in which  $6 \neq 9$ ,  $6 \neq 1$ ,  $3 \neq 1$ ,  $9 \neq 1$  (so that we have full rank specified submatrices) and  $8 \neq 10$ . As an aside, we note that for some of the small fields it may be impossible to fulfill the nondegeneracy requirement on the data. E.g., when  $\mathbb{F} = \{0, 1\}$  a  $2 \times 2$  matrix can only be nonsingular if zeroes are allowed in the matrix.

In order to formulate our results and conjectures it is useful to introduce the *bipartite graph* associated with a pattern. Given a pattern  $J$  the corresponding (undirected) bipartite graph  $G(J)$  has vertices  $\{v_1, \dots, v_n, u_1, \dots, u_m\}$ , and  $(v_i, u_j)$  is an edge in  $G(J)$  if and only if  $(i, j) \in J$ . A subset  $K \subseteq J$  is called a *cycle* in  $G(J)$  if  $K$  is of the form

$$K = \{(i_1, j_1), (i_2, j_1), (i_2, j_2), \dots, (i_k, j_k), (i_1, j_k)\} \quad (3.4)$$

for distinct  $i_1, \dots, i_k$  and distinct  $j_1, \dots, j_k$ , where  $k \geq 2$ . In other words,  $G(K)$  is a cycle in  $G(J)$ . The *length* of the cycle  $K$  is its number of elements; so for  $K$  in (3.4) we have that its length is  $2k$ . Note that in a bipartite graph all cycles have even length. When a pattern is 1-sparse, it cannot have any cycles as a cycle of length  $k$  requires the corresponding partial matrix to have  $2k$  specified entries in a  $k \times k$  submatrix. This observation makes it easy to apply a result in [5] to prove the statement by Ingerman and Strang.

**Proposition 3.2** [11] *Any partial matrix whose pattern is 1-sparse and whose given entries are nonzero, has a rank 1 completion.*

**Proof.** Let  $J$  denote the pattern of specified entries of the matrix. By the observation before the proposition we have that  $G(J)$  does not contain any cycles. Therefore by Lemma 6.2 in [5] it suffices to observe that the absence of zero specified entries implies trivially that the partial matrix "is singular with respect to all three lines in  $G(J)$ " (the latter condition can be rephrased as saying that for all  $2 \times 2$  submatrices there is a rank one completion).  $\square$

We say that the cycle  $K$  in (3.4) has a *chord* in  $J$  if for some  $1 \leq p, q \leq k$  we have that  $(i_p, j_q) \in J \setminus K$ . The cycle  $K$  is called *minimal* in  $J$  if it does not have a chord in  $J$ . A bipartite graph is called *chordal* if it does not have minimal cycles of length 6 or larger.

Notice that the bipartite graph associated with the partial matrix in Example 3.1 is a minimal cycle of length 8, and therefore the graph is not chordal. Therefore it could be that the obstruction of finding a solution in Example 3.1, lies in the non-chordality of the underlying bipartite graph. As we have seen in [5, Theorem 3.1] it is also the non-chordality that prevents graphs from being so called "rank determined". In addition, notice that in the  $r = 1$  case the 1-sparse property prevents the existence of minimal cycles of length 6 or more, and thus in that case all patterns are automatically bipartite chordal. Thus we arrive at the following conjecture.

**Conjecture 3.3** *Consider a  $r$ -sparse partial matrix  $\mathcal{A}$  with a bipartite graph that is chordal. If all fully specified submatrices have full rank, then  $\mathcal{A}$  has a completion of rank at most  $r$ .*

We can prove the conjecture for the subclass of banded patterns. Recall (cf. [25]) that a pattern  $J \subset \{1, \dots, n\} \times \{1, \dots, m\}$  is called *banded* if there exist permutations  $\sigma$  on  $\{1, \dots, n\}$  and  $\tau$  on  $\{1, \dots, m\}$  so that

$$J_{\sigma, \tau} := \{(\sigma(i), \tau(j)) ; (i, j) \in J\}$$

satisfies

$$(i, j), (k, l) \in J_{\sigma, \tau}, i \leq k, j \geq l \Rightarrow \{i, \dots, k\} \times \{l, \dots, j\} \subset J_{\sigma, \tau}.$$

**Theorem 3.4** *Let  $\mathbb{F}$  be an infinite subfield of  $\mathbb{C}$ . Consider a  $r$ -sparse partial matrix with a banded pattern and suppose that all fully specified submatrices have full rank. Then there exists a completion of rank at most  $r$ .*

To prove this result we need the notion of triangular pattern: a pattern  $J$  is called *triangular* if there exist permutations  $\sigma$  and  $\tau$  so that  $J_{\sigma,\tau}$  satisfies

$$(i, j) \in J_{\sigma,\tau} \Rightarrow \{i, \dots, n\} \times \{1, \dots, j\} \subset J_{\sigma,\tau}.$$

*Proof of Theorem 3.4.* By Theorem 1.1 in [25] it suffices to show that for every triangular subpattern (for the definition, see [25]) we have that the minimal rank is  $\leq r$ . But a triangular subpattern can always be embedded in a pattern that corresponds to  $r$  rows and columns specified (due to the condition that in any  $k \times k$  submatrix at most  $(2k - r)r$  entries are specified). But then the result follows.  $\square$

Observe that the proof shows that if the bipartite chordal minimal rank conjecture (Conjecture 3.3 in [5]; see also Chapter 5 in [23]) is true, then Conjecture 3.3 above is true as well. The techniques developed in [2] and/or [12] may be helpful in proving this conjecture above.

While Example 3.1 shows that not every  $r$ -sparse partial matrix with full rank specified submatrices has a rank  $r$  completion, it should be noticed that if we consider the example as one over an infinite subfield of  $\mathbb{C}$  and we perturb the data slightly, then there is a rank  $r$  completion. This observation leads to the following.

Let  $\mathbb{F}$  be an infinite subfield of  $\mathbb{C}$ , and let  $J$  be a pattern. Consider the set  $\mathcal{P}_J$  of partial matrices over  $\mathbb{F}$  with pattern  $J$ . We can identify  $\mathcal{P}_J$  with the set  $\mathbb{F}^{|J|}$ , and we use this correspondence and the usual topology on  $\mathbb{F}^{|J|}$  to define a topology on  $\mathcal{P}_J$ . We can now formulate the following conjecture, which probably best describes the conjecture that Ingerman and Strang had in mind.

**Conjecture 3.5** *Let  $J$  be an  $r$ -sparse pattern. Then there is a dense subset  $\mathcal{P}'$  of  $\mathcal{P}_J$ , so that all partial matrices in  $\mathcal{P}'$  have a completion of rank at most  $r$ .*

Analyzing Example 3.1 it is not hard to convince oneself that Conjecture 3.5 is true for the 2-sparse pattern  $J = \{1, \dots, 4\} \times \{1, \dots, 4\} \setminus \{(1, 4), (2, 3), (3, 2), (4, 1)\}$ . Indeed, a desired set  $\mathcal{P}'$  is described by all partial matrices for which the given entries satisfy 4 nonequalities. These nonequalities are obtained by eliminating three out of four unknowns, and requiring that the coefficient in front of the highest power is not equal to 0; doing this for all possible choices of the remaining unknown, we get 4 nonequalities. Also, by the results stated earlier, the conjecture is true for  $r = 1$ , and for banded  $J$ .

One may perhaps prove Conjecture 3.5 in the following way. View the partial matrices in  $\mathcal{P}_J$  as a regular matrix where the unknowns are represented by variables  $x_1, \dots, x_k$ ,  $k = nm - |J|$ . Let  $I$  be the ideal generated by the determinants  $p_1, \dots, p_N$

of  $(r + 1) \times (r + 1)$  submatrices as polynomials in the unknowns  $x_1, \dots, x_k$ . We now need to show that generically, the constant polynomial 1 is not in  $I$ . Use of elimination theory may perhaps be used to now show that one can only eliminate all variables when the coefficients satisfy certain equalities. Finally, observing that these equalities are generically not satisfied one may finish the proof.

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