

DEPARTMENT OF MATHEMATICS
DREXEL UNIVERSITY
PH.D. QUALIFYING EXAMINATION
JUNE 14, 2010

Instructions:

- The exam consists of six problems which are equally weighted. The time for examination is $3\frac{1}{2}$ hours.
 - Do 4 out of 6 of the analysis questions in Section 1.
 - Do 2 out of 3 of the linear algebra questions in Section 2.
 - Indicate clearly which of your questions are to be graded. If you do **not** indicate which of your questions are to be graded, the default will be to grade questions one through four of the analysis section and questions one and two of the linear algebra section.
 - Please ask the proctor about any obvious typographic errors.
 - Along with this list of problems, you will be given two examination notebooks. Use one of them for presenting your solutions. The other one may be used for auxiliary calculations. Both notebooks must be submitted when the exam is over.
 - Every solution should be given a concise but sufficient explanation and written up legibly. Try to keep a one inch margin on the papers.
 - This is a closed book exam.
 - No electronic devices are allowed.
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Remember: you are to answer 4 out of the following 6 Analysis problems.

- (1) A mapping $f : X \rightarrow Y$ is called an *open* mapping between two metric spaces X and Y if the image of every open set of X is open in Y .
- (a) If $f : \mathbb{R} \rightarrow \mathbb{R}$ is an open and continuous mapping, prove that f must be a monotone function.
- (b) Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be an open and continuous mapping. Let D denote the closed unit disk. Prove that the maximum of f on D must occur on the boundary of the unit disk D .
- (2) (a) Let $a \in \mathbb{R}$ and let f be a twice-differentiable function on (a, ∞) . Let M_0, M_1, M_2 be the least upper bounds of $|f(x)|, |f'(x)|, |f''(x)|$, respectively, on (a, ∞) . Prove that

$$M_1^2 \leq 4M_0M_2.$$

(b) Suppose f is twice-differentiable on $(0, \infty)$, f'' is bounded on $(0, \infty)$, and $f(x) \rightarrow 0$ as $x \rightarrow \infty$. Prove that $f'(x) \rightarrow 0$ as $x \rightarrow \infty$.

- (3) (a) Let f be a bounded Lebesgue measurable function on the interval $[a, b]$. Let $\int_a^b f(x) dx$ and $\int_a^b f(x) dx$ denote the lower and upper Riemann integrals of f , respectively. Let $\int_{[a,b]} f dm$ denote the Lebesgue integral of f . Prove that

$$\int_a^b f(x) dx \leq \int_{[a,b]} f dm \leq \int_a^b f(x) dx.$$

(b) Give an example to show that the value of Lebesgue integral can be *strictly* between the values of the lower and upper Riemann integrals of f .

- (4) (a) Let $f : [0, 1] \rightarrow \mathbb{R}$ be continuous with $f(0) = 0$. Prove that there is a sequence of polynomials $\{Q_n(x)\}$ such that $xQ_n(x) \rightarrow f(x)$ uniformly on $[0, 1]$.
- (b) Let $\{g_n(x)\}$ be a sequence of non-negative continuous functions on $[a, b]$ so that $g_n(x) \geq g_{n+1}(x)$ and $\{g_n(x)\}$ converges pointwise to zero. Let $\varepsilon > 0$ be given and let

$$E_n = \{x \in [a, b] : g_n(x) \geq \varepsilon\}.$$

Prove that there exists an integer N so that $E_N = \emptyset$.

- (5) Let $M_2(\mathbb{R})$ denote the space of all 2×2 real matrices. Let $T : M_2(\mathbb{R}) \rightarrow M_2(\mathbb{R})$ by

$$A \mapsto A^2.$$

(a) Prove that T is differentiable and that its derivative is given by

$$T'(A)(H) = AH + HA, \quad H \in M_2(\mathbb{R}).$$

(b) Prove that there is a continuously differentiable function S defined in a neighborhood U of the identity I in $M_2(\mathbb{R})$ such that

$$S \circ T(A) = A, \quad \text{for all } A \in U.$$

- (6) (a) Let $\{a_n\}$ be a sequence of real numbers such that $\sum_{k=1}^{\infty} |a_k|$ converges. Prove that $\{a_k\}$ is a bounded sequence. Prove that for any real number $q > 1$, the series $\sum_{k=1}^{\infty} |a_k|^q$ converges.
- (b) Let $\{a_n\}$ be a sequence of real numbers such that $\sum_{k=1}^{\infty} |a_k|$ converges. For every natural number n , let $f_n(x) : [0, 1] \rightarrow [-3, 7]$ be a continuous function. Prove that $\sum_{k=1}^{\infty} a_k f_k(x)$ converges; call the sum $f(x)$. Prove that $f(x)$ is a continuous function.

2. LINEAR ALGEBRA

Remember: you are to answer 2 out of the following 3 Linear Algebra problems.

- (1) (a) Prove that if x is the eigenvector of a normal matrix N corresponding to the eigenvalue λ then x is also the eigenvector of the adjoint matrix N^* corresponding to the eigenvalue $\bar{\lambda}$.
(b) Prove that the eigenvectors of a normal matrix N corresponding to two distinct eigenvalues are orthogonal.
(c) Let e be an eigenvector of a normal matrix N . Show that the subspace L consisting of all vectors orthogonal to e is invariant for N (i.e., $Nx \in L$ for every $x \in L$).
(d) Prove that a square matrix N is normal if and only if every eigenvector of N is also an eigenvector of N^* .
- (2) (a) Find the Jordan canonical form (up to a permutation of diagonal blocks) of the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

- (b) Find the singular value decomposition of the matrix

$$A = \begin{bmatrix} 1 & -3 \\ 3 & -1 \end{bmatrix}.$$

- (3) (a) For which $\alpha_1, \dots, \alpha_n$ the $(n+1) \times (n+1)$ matrix

$$A = \begin{bmatrix} 0 & \alpha_1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \alpha_n \\ 0 & \dots & \dots & \dots & 0 \end{bmatrix}$$

is similar to a diagonal matrix?

- (b) Find a nonsingular matrix T such that $B = T^{-1}AT$ is a diagonal matrix, and find this matrix B .
Here

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

- (c) Generalize the results in part (b) to the case of a $n \times n$ matrix

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}.$$