Department of Mathematics
Drexel University
Ph.D. Qualifying Examination
September 14, 2010

Instructions:

- The exam consists of six problems which are equally weighted.
  The time for examination is $\mathbf{3\frac{1}{2}}$ hours.
- Do **4** out of 6 of the analysis questions in Section 1.
- Do **2** out of 3 of the linear algebra questions in Section 2.
- Indicate clearly which of your questions are to be graded. If you do **not** indicate which of your questions are to be graded, the default will be to grade questions one through four of the analysis section and questions one and two of the linear algebra section.
- Please ask the proctor about any obvious typographic errors.
- Along with this list of problems, you will be given two examination notebooks. Use one of them for presenting your solutions. The other one may be used for auxiliary calculations. Both notebooks must be submitted when the exam is over.
- Every solution should be given a concise but sufficient explanation and written up legibly. Try to keep a one inch margin on the papers.
- This is a closed book exam.
- No electronic devices are allowed.

Remember: you are to answer 4 out of the following 6 Analysis problems.

(1) (a) Construct a bijection T (that is, a map which is both 1-1 and onto) from the open interval $(0,1)$ onto the closed interval $[0,1]$.
(b) Prove or disprove: it is possible that T in part (a) can be continuous.

(2) (a) Let $f:\mathbb{R}\to\mathbb{R}$ be differentiable everywhere. Suppose that $f(0)=0$ and that $|f'(x)|\le|f(x)|$ for all x. Prove that $f(x)=0$ for all x.
(b) Let $\{a_n\}$ be a sequence of positive real numbers. Assume that

$$\left(\limsup_{n\to\infty} a_n\right)\left(\limsup_{n\to\infty}\frac{1}{a_n}\right)=1,$$

prove that $\{a_n\}$ is a convergent sequence.

(3) (a) Let A be a countably infinite subset of $(0,1)$. Construct a monotone increasing function f on $[0,1]$ such that $f(0)=0$, $f(1)=1$, and f is continuous at $x\in[0,1]$ if and only if $x\notin A$.
(b) Prove or disprove that the function f constructed in part (a) is Riemann integrable.

(4) Let $f_n=\begin{cases} n, & 0\le x\le\frac{1}{n},\\ 0, & \text{otherwise}.\end{cases}$ Let $\phi\in C^\infty(\mathbb{R})$.
(a) Show that $\displaystyle\int_{-\infty}^{\infty}\sqrt{f_n(x)}\phi(x)\,dx\to 0$ as $n\to\infty$.
(b) Show that $\displaystyle\int_{-\infty}^{\infty}f_n(x)\phi(x)\,dx\to\phi(0)$ as $n\to\infty$.

(5) Let a_n, with $n=1,2,3,\ldots$, be a sequence of real numbers. Let $S_n=\displaystyle\sum_{k=1}^{n}a_n$, and let $\sigma_n=\displaystyle\frac{1}{n}\sum_{k=1}^{n}S_k$.

Part (a) is not correctly stated.

(a) Show that if S_n is a bounded sequence, then σ_n converges as $n\to\infty$.
(b) Show that if $S_n\to S$ as $n\to\infty$, then $\sigma_n\to S$ also.

(6) (a) Let $f:\mathbb{R}^2\to\mathbb{R}$ be given by

$$f(x,y)=\begin{cases} 0, & (x,y)=(0,0)\\ \dfrac{xy}{x^2+y^2}, & (x,y)\ne(0,0)\end{cases}.$$

Prove that f is *not* differentiable at $(0,0)$.
(b) Consider the two equations

$$\begin{cases} xy^2+xzu+yv^2=3,\\ u^3yz+2xv-u^2v^2=2.\end{cases}$$

Is it possible to solve these equations for u and v as functions of (x,y,z) in a neighborhood of the points $(x,y,z)=(1,1,1)$ and $(u,v)=(1,1)$? Prove your answer.

Remember: you are to answer 2 out of the following 3 Linear Algebra problems.

(1) (a) Show that two Hermitian $n \times n$ matrices A and B are similar if and only if they are unitarily similar.

(b) For two Hermitian $n \times n$ matrices A and B, show that

$$\mathrm{trace}\{(AB)^2\} \leq \mathrm{trace}\{A^2B^2\}.$$

Hint: Show that $AB - BA$ is skew-Hermitian and consider $\mathrm{trace}\{(AB - BA)^2\}$.

(2) Find the Jordan canonical form (up to a permutation of diagonal blocks) of the matrix A if

(a) $A = \begin{bmatrix} 12 & -6 & -2 \\ 18 & -9 & -3 \\ 18 & -9 & -3 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 1 & \dots & \dots & 1 \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 \end{bmatrix}$, where A is of size $n \times n$.

(c) $A = \begin{bmatrix} 0 & \alpha & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \alpha \\ \alpha & 0 & \dots & \dots & 0 \end{bmatrix}$, where A is of size $n \times n$, and α some complex number.

(3) For two $n \times n$ matrices A and B, show that the characteristic polynomials of AB and BA are equal. **Hint:** Show first that if at least one of matrices A and B is non-singular, then AB and BA are similar.