DEPARTMENT OF MATHEMATICS
DREXEL UNIVERSITY
PH.D. QUALIFYING EXAMINATION
JUNE 26, 2008

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**Instructions**:

- The exam consists of six problems which are equally weighted.
  The time for examination is $\mathbf{3\frac{1}{2}}$ hours.
- Do **4** out of 6 of the analysis questions in Section 1.
- Do **2** out of 3 of the linear algebra questions in Section 2.
- Indicate clearly which of your questions are to be graded. If you do **not** indicate which of your questions are to be graded, the default will be to grade questions one through four of the analysis section and questions one and two of the linear algebra section.
- Please ask the proctor about any obvious typographic errors.
- Along with this list of problems, you will be given two examination notebooks. Use one of them for presenting your solutions. The other one may be used for auxiliary calculations. Both notebooks must be submitted when the exam is over.
- Every solution should be given a concise but sufficient explanation and written up legibly. Try to keep a one inch margin on the papers.
- This is a closed book exam.
- No electronic devices are allowed.

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**Remember: you are to answer 4 out of the following 6 Analysis problems.**

(1) Let $f : \mathbb{R} \to \mathbb{R}$ be $n$-times differentiable, and suppose that the $n$-th derivative $f^{(n)}$ is an increasing function. Given that $f^{(k)}(0) = 0$ for $0 \le k \le n$ and that $f(1) = 1$, show that
  (a) $f^{(n)}(1) \ge n!$,
  (b) $f^{(k)}(1) \ge 0$, for $1 \le k \le n-1$,
  (c) $f(3) \ge 2^n$.

(2) (a) Prove that if $f$ and $g$ are real-valued functions on a metric space $(M,d)$ which are both bounded and uniformly continuous, then their product $fg$ is also bounded and uniformly continuous on $M$.
(b) Let $f(x) = x$ and $g(x) = \sin x$. Show that they are both uniformly continuous on $\mathbb{R}$, however their product $(fg)(x) = x\sin x$ is not uniformly continuous on $\mathbb{R}$.

(3) Consider the series $\sum_{n=1}^{\infty} f_n(x)$ on $[0,1]$ where

$$f_n(x) = \begin{cases} 0, & \text{if } 0 \le x \le 2^{-(n+1)}, \\ \frac{1}{n}\sin^2(2^{n+1}\pi x), & \text{if } 2^{-(n+1)} < x < 2^{-n}, \\ 0, & \text{if } 2^{-n} \le x \le 1. \end{cases}$$

(a) Find the pointwise sum of the series.
(b) Prove that the series converges uniformly on $[0,1]$.
(c) Prove that the series cannot be majorized by a convergent series with non-negative terms; that is, show that

$$\sum_{n=1}^{\infty} \left( \sup_{x\in[0,1]} |f_n(x)| \right) = +\infty.$$

(4) (a) Find the cardinality of the set of all convergent series of the form $\sum_{n=1}^{\infty} a_n$, where $a_n$ is the reciporcal of a positive integer; that is, $1/a_n \in \mathbb{Z}^+$ for all $n$. Prove your answer.
(b) Let $A \subset \mathbb{R}^k$ have the following property: every $a \in A$ has a neighborhood $N_r(a)$, for some $r > 0$, such that $N_r(a) \cap A$ is countable. Prove that $A$ is countable.

(5) Evaluate the following two limits. For full credit, show all details.

$$(a) \qquad \lim_{n\to\infty} \int_0^{\pi} \frac{n + \sin(nx)}{3n + \sin^2(nx)}\, dx,$$

$$(b) \qquad \lim_{n\to\infty} \int_0^{n} \frac{1}{\sqrt{x}} \frac{1}{\sqrt{1+n^2x^2}} \cos(nx)\, dx.$$

*Hint*: for part (b), you may want to consider the integral over $[0,1]$ and $[1,n]$ separately.

(6) (a) Consider the two equations

$$\begin{cases} xy^2 + xzu + yv^2 = 3, \\ u^3yz + 2xv - u^2v^2 = 2. \end{cases}$$

Is it possible to solve these equations for $u$ and $v$ as functions of $(x,y,z)$ in a neighborhood of the points $(x,y,z) = (1,1,1)$ and $(u,v) = (1,1)$? Prove your answer.
(b) Suppose that $T : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable and its differential $T'(x)$ is invertible for all $x \in \mathbb{R}^n$. Prove that the image of any open subset of $\mathbb{R}^n$ under $T$ is open.

**Remember: you are to answer 2 out of the following 3 Linear Algebra problems.**

(1) Determine whether one can find matrices with the following properties. If so, provide such a matrix; if not explain why not

(a) a $3 \times 3$ matrix with all its eigenvalues equal to 0, and with singular values $1, \frac{1}{2}, 0$

(b) a $3 \times 3$ matrix with all its entries positive real numbers and with eigenvalues $1, 1, \frac{1}{2}$

(c) a $3 \times 3$ unitary matrix with singular values $1, 1, i$

(d) a $3 \times 3$ real matrix with eigenvalues $3, 2+i, 2-i$

(2) Consider the tri-diagonal Hermitian matrix

$$A = \begin{pmatrix} a_1 & \overline{b_2} & & 0 \\ b_2 & \ddots & \ddots & \\ & \ddots & \ddots & \overline{b_n} \\ 0 & & b_n & a_n \end{pmatrix}$$

$a_i \in \mathbb{R}, i = 1, \ldots, n$.

(a) Show that the eigenvalues $\lambda_j, j = 1, \ldots, n$, of $A$ satisfy $|\lambda_j| \leq \max_{1 \leq i \leq n} \{|b_i| + |a_i| + |b_{i+1}|\}$, where $b_1 = b_{n+1} = 0$.

The eigenvalues of $A$ are numerically typically computed via the $QR$ algorithm, which follows the following iteration

$$A_0 = A$$

if $A_i = Q_i R_i, Q^* Q = I$, $R$ upper triangular, then put

$$A_{i+1} = R_i Q_i$$

(b) Explain why $A_i$ and $A_{i+1}$ have the same eigenvalues.

(c) Explain why all $A_i$'s are tri-diagonal.

(3) (a) Let $p(x) = (x-2)^2(x-3)^2$. Determine a matrix $A$ for which $p(A) = 0$ and for which $q(A) \neq 0$ for all nonzero polynomials $q$ of degree $\leq 3$. Explain why $q(A) \neq 0$ for such $q$.

(b) Determine the Jordan canonical form of $\begin{pmatrix} 0 & 2 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

(c) Show that if $A$ and $B$ are square matrices, with $A$ invertible, then $AB$ and $BA$ have the same Jordan canonical form.