

DEPARTMENT OF MATHEMATICS  
DREXEL UNIVERSITY  
PH.D. QUALIFYING EXAMINATION  
JUNE 28, 2007

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**Instructions:**

- The exam consists of six problems which are equally weighted.  
The time for examination is 3 1/2 hours.
  - Do 4 out of 6 of the analysis questions in Section 1.
  - Do 2 out of 3 of the linear algebra questions in Section 2.
  - Indicate clearly which of your questions are to be graded. If you do **not** indicate which of your questions are to be graded, the default will be to grade questions one through four of the analysis section and questions one and two of the linear algebra section.
  - Please ask the proctor about any obvious typographic errors.
  - Along with this list of problems, you will be given two examination notebooks. Use one of them for presenting your solutions. The other one may be used for auxiliary calculations. Both notebooks must be submitted when the exam is over.
  - Every solution should be given a concise but sufficient explanation and written up legibly.  
Try to keep a one inch margin on the papers.
  - This is a closed book exam.
  - No electronic devices are allowed.
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# 1. ANALYSIS

- (1) (a) Consider a sequence of functions  $f_n : [a, b] \rightarrow \mathbb{R}$  with the property that for each  $x \in [a, b]$  there is an open interval  $I_x$  such that the sequence  $\{f_n\}$  converges uniformly on  $I_x \cap [a, b]$ . Prove that  $\{f_n\}$  converges uniformly on the entire interval  $[a, b]$ .  
 (b) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be differentiable everywhere. Suppose that  $f(0) = 0$  and that  $|f'(x)| \leq |f(x)|$  for all  $x$ . Prove that  $f(x) = 0$  for all  $x$ .
- (2) (a) Give an example of a sequence  $\{x_n\}$  in  $\mathbb{R}$  so that  $\lim_{n \rightarrow \infty} |x_n - x_{n+1}| = 0$  but the sequence does not converge.  
 (b) Prove that if  $|x_n - x_{n+1}| < 2^{-n}$ , then  $\lim_{n \rightarrow \infty} x_n$  does exist.
- (3) (a) Let  $f$  be a bounded function on the compact interval  $[a, b]$ . Assume that  $f$  is Riemann integrable on every subinterval  $[\alpha, \beta]$  where  $a < \alpha < \beta < b$ . Prove that  $f$  is Riemann integrable on  $[a, b]$ .  
 (b) Let  $f$  be any function on the compact interval  $[a, b]$  such that  $f$  is Riemann integrable on every subinterval  $[\alpha, \beta]$  where  $a < \alpha < \beta < b$ . Prove or disprove by giving a counterexample that  $f$  is Riemann integrable on  $[a, b]$ .
- (4) Consider the integral  $\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx$  where  $0 < a < b$ . Justify carefully all steps in the following parts.  
 (a) Show that  $\int_1^\infty [e^{-ax} - e^{-bx}]/x dx = \int_a^b e^{-u}/u du$ .  
 (b) Show that  $\int_0^1 [e^{-ax} - e^{-bx}]/x dx = -\int_a^b e^{-u}/u du + \ln(b/a)$ . Finally, evaluate  $\int_0^\infty [e^{-ax} - e^{-bx}]/x dx$ .
- (5) Let  $\phi(x)$  be a function defined for all  $x > 0$ . Suppose that for sufficiently large  $x$ , the function can be represented by a convergent series
- $$\phi(x) = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_3}{x^3} + \cdots$$
- where  $a_0, a_1, a_2, \dots$  are all real. Prove that the series
- $$\phi(1) + \phi(2) + \phi(3) + \cdots$$
- converges if and only if  $a_0 = 0$  and  $a_1 = 0$ .
- (6) Consider the mapping  $S$  from the space of  $2 \times 2$  real matrices back into itself given by  $S(A) = A^2$ .  
 (a) Find the differential of  $S$  at the matrix  $A$ .  
 (b) Observe that  $S(-I) = I$ . Does there exist an inverse mapping, that is, a mapping  $g$  such that  $S(g(A)) = A$ , defined in a neighborhood of the identity  $I$  such that  $g(I) = -I$ ? Prove your answer.

## 2. LINEAR ALGEBRA

- (1) Let  $A$  be a  $6 \times 6$  matrix so that  $A^6 = 0$  and its rank is 3.
- (a) Determine the possible Jordan canonical forms of  $A$ .
  - (b) If, in addition, it is given that for some vector  $x$  we have that  $A^2x \neq 0$ , what can you say about the Jordan canonical form of  $A$ .
  - (c) Determine the Jordan canonical form of the matrix
 
$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ -3 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$
  - (d) True or false: matrices with the same Jordan canonical form have the same eigenvectors. If true, explain briefly. If false, provide a counterexample.
- (2) Let  $A$  be a  $4 \times 4$  matrix with eigenvalues  $1, 2i, 2i, 3$  and singular values  $s_1 \geq s_2 \geq s_3 \geq s_4$ .
- (a) Determine the product  $s_1 s_2 s_3 s_4$ .
  - (b) Show that  $s_1 \geq 3$ .
  - (c) Can it happen that  $s_4 < 1$ ? Explain your answer.
  - (d) Express  $\text{trace}(A^*A + A)$  in terms of  $s_1, s_2, s_3$  and  $s_4$ .
  - (e) Assuming that  $A$  is normal, determine  $s_1 + s_2 + s_3 + s_4$ .
- (3) Let  $A = [a_{ij}]_{i,j=1}^n$  be a matrix so that  $a_{ij} \geq 0$  for all  $i$  and  $j$ , and so that  $\sum_{k=1}^n a_{ik} = 2$  for all  $i = 1, 2, \dots, n$ .
- (a) Show that 2 is an eigenvalue for  $A$ .
  - (b) Argue that if  $\lambda$  is an eigenvalue for  $A$ , then  $|\lambda| \leq 2$ .
  - (c) Can it happen that  $A$  has an eigenvalue  $\lambda$  with  $|\lambda| = 2$  and  $\lambda \neq 2$ ? If not, explain. If yes, give a counterexample.
  - (d) Let, in addition, be given that  $\text{tr}(A) = 2n$ . Determine the possible matrices  $A$  can be equal to.