DEPARTMENT OF MATHEMATICS
DREXEL UNIVERSITY
PH.D. QUALIFYING EXAMINATION

June 9, 2005

Name ____

About this exam:
Below are six questions, each worth 15 points. There is one extra credit problem worth 15 points. Ten base points will be added, hence the total point value of the exam is 100 points, and the maximum score possible if you do the extra credit is 115.

All solutions require proofs. No points will be given for "yes/no" answers.

This is a closed book exam. No electronic devices are allowed.

The exam is 21 pages long.

When the announcement is made that the exam is over, STOP writing immediately.

1. Consider the metric space of rational numbers $\mathbb{Q}$ with the usual absolute value metric. Fix irrational numbers a and b and let S be the set of rational numbers in the open interval (a, b).

a. (7 points) Show that S is closed and bounded as a subset of $\mathbb{Q}$.

1b. (8 points) Is S is compact as a subset of $\mathbb{Q}$? Prove your answer.

2. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and assume that there exists at least one point x_0 at which f is continuous. Suppose that f satisfies the equations

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$. The goal of this problem is to show that there is a real number a such that $f(x) = ax$.

a. (5 points) Show that f is continuous at every point.

2b. (5 points) Show that for any rational number p/q and real number x that

$$f(\frac{p}{q}x) = \frac{p}{q}f(x).$$

2c. (5 points) Show that there is a real number a such that $f(x) = ax$.

3. Let f be continuous with

$$\int_0^1 f(x)x^n dx = 0, n = 0, 1, 2, ...$$

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In this problem you will show that $f(x) = 0$ for all $x \in [0, 1]$. Note that you can do part (d) without having done (b) or (c).

3a.(4 points) Show that for every polynomial p we have that

$$\int_0^1 f(x)p(x)dx = 0.$$

3b. (4 points) State precisely the theorem of Weierstrass (or Stone-Weierstrass) on the approximation of a function by polynomials.

3c. (4 points) Use the theorem that you stated in (b) to show that

$$\int_0^1 f(x)^2 dx = 0.$$

3d. (3 points) Use (c) to show that $f(x) = 0$ for $x \in [0, 1]$.

4. Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$. Call x a fixed point if $f(x) = x$.

a.(5 points) Prove that if f is differentiable and $f'(t) \neq 1$ for all real t, that f has at most one fixed point.

4b.(5 points) Show that the function defined by

$$f(t) = t + (1 + e^t)^{-1}$$

has not fixed point, and that $0 < f'(t) < 1$ for all real t.

4c.(5 points) Show that if a constant $A < 1$ exists such that $|f'(t)| < A$ then a fixed point x of f exists, and that $x = \lim x_n$ where x_1 is an arbitrary real number and

$$x_{n+1} = f(x_n)$$

for $n = 1, 2, 3, ...$

5a. (5 points) State precisely the implicit function theorem.

5b.(5 points) Consider the system of non-linear equations

$$\begin{aligned} 3x + y - z + u^2 &= 0 \\ x - y + 2z + u &= 0 \\ 2x + 2y - 3z + 2u &= 0 \end{aligned}$$

Show that this system can be solved for x, y, u in terms of z.

5c.(5 points) Show that it is not possible to solve the system in 5b for x, y, z in terms of u.

6. Let

$$f(x,y) = \begin{cases} 0 & (x,y) = (0,0) \\ \frac{x^2 y}{x^4+y^2} & \text{otherwise} \end{cases}$$

a. (4 points) Show that f is discontinuous at $(0,0)$.

6b. (4 points) Show that $\frac{\partial f}{\partial x}(x, y)$ exists for every $(x, y) \in \mathbb{R}^2$.

6c. (4 points) Give the definition of what it means for a function $g(x, y)$ to be differentiable at $(0, 0)$.

6d. (3 points) For your information,

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} 0 & (x,y) = (0,0) \\ \frac{x^6 - x^2y^2}{(x^4+y^2)^2} & \text{otherwise} \end{cases}$$

Does the above example show that functions of two variables do not have to be continuous to be differentiable ? Explain.

7. (Extra credit - 15 points) Show that any uncountable subset of $\mathbb{R}^k$ has a limit point.